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LETTER TO THE EDITOR

Gowdy $S^1 \otimes S^2$ and S^3 inhomogeneous cosmological models

J-L Hanquin† and J Demaret

Institut d'Astrophysique, Université de Liège, B 4200 Cointe-Ougree, Belgium

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Abstract. The ADM-type Hamiltonian formulation of Gowdy $S^1 \otimes S^2$ and S^3 vacuum inhomogeneous cosmological models is given. The canonical and constraint equations are explicitly solved and the asymptotic behaviour of the most general solution in the neighbourhood of the initial and final inhomogeneous scalar singularities is shown to be of the generalised Kasner type.

Gowdy's closed empty cosmological models, which can be thought of as inhomogeneous models filled with gravitational waves locally indistinguishable from Einstein–Rosen cylindrical waves (cf Gowdy, 1971, 1974), belong to three different types following the topology of their compact space sections which can be of the 3-torus ($S^1 \otimes S^1 \otimes S^1$), the three-handle ($S^1 \otimes S^2$) or the 3-sphere ($S^3$) type.

Although the simplest model—with the 3-torus topology—has been intensively used in recent cosmological work, mostly for the purpose of studying particle creation in the early universe (cf Berger 1972, Misner 1973), the two other types of models do not seem to have received much attention from the cosmologists. However, they possess very interesting characteristics which could be fruitfully used in the framework of research work connected with the design of more general models than the Friedmann ones to represent the geometry of the early universe. More precisely, the Gowdy S^3 model appears as an empty inhomogeneous generalisation of the closed Friedmann models (whose space-like sections have the same topology), and both S^3 and $S^1 \otimes S^2$ models possess an initial as well as a final singularity, whose nature it is interesting to elucidate.

We give here the principal results of a detailed study of the fundamental properties of Gowdy $S^1 \otimes S^2$ and S^3 models based on ADM-type Hamiltonian methods, which lead to a reduced ADM-Hamiltonian suitable for studies of the canonical quantisation of the models and of the phenomenon of creation of particles in the framework of the usual semiclassical approach initiated by Parker (1966). Moreover, the resolution of the canonical and constraint equations enables one to obtain exact analytic solutions for the classical models and to discuss the nature of their singularities.

The metric of Gowdy $S^1 \otimes S^2$ and S^3 space–times (characterised by two mutually orthogonal space-like Killing vector fields) has the form (cf Gowdy, 1971, 1974)

$$ds^2/L^2 = e^A(d\theta^2 - dt^2) + R(e^{X+Y}\, d\sigma^2 + e^{-X-Y}\, d\delta^2). \tag{1}$$

The angles θ and δ describe the two-spheres in the case of the three-handle $S^1 \otimes S^2$ models, while in the case of S^3 models, the angles θ, $\sigma - \delta$ and $\sigma + \delta$ are the Euler angles characterising the 3-spheres.

† Boursier IRSAI.

The functions A, R, X and Y depend only on coordinates t and θ. L is a constant length so chosen that the maximum value of R is equal to 1; in the following, L will be put equal to 1. The ranges of variation of the variables, for both Gowdy models, are

$$t \in (0, \pi), \qquad \theta \in [0, \pi], \qquad \sigma \text{ and } \delta \in [0, 2\pi].$$

The difference between the (0, 0) and (1, 1) components of Einstein's vacuum field equations, for the metric (1), gives the following result:

$$R'' - \ddot{R} = 0 \tag{2}$$

(the dots and primes denote respectively differentiation with respect to t and θ).

The solution $R = \sin\theta \sin t$ leads to the $S^1 \otimes S^2$ and S^3 Gowdy models (a series of matching conditions is then necessary to ensure the validity of Einstein's field equations for $\theta = t$ and $\theta + t = \pi$ (cf Gowdy, 1971, 1974)). Y is chosen so as to express the metric of 3-spaces of Gowdy's models as closely as possible in terms of the usual metrics for 3-handles and 3-spheres, i.e.

$$Y = -\ln R \qquad \text{for the } S^1 \otimes S^2 \text{ topology,} \tag{3a}$$

$$Y = -\ln \tan \theta/2 \qquad \text{for the } S^3 \text{ topology.} \tag{3b}$$

Moreover, to avoid conical singularities at $\theta = 0$ and $\theta = \pi$, one has to impose the following conditions on the functions A and X:

$$A'(t, 0) = A'(t, \pi) = 0, \qquad X'(t, 0) = X'(t, \pi) = 0, \tag{4', 4''}$$

$$\begin{cases} X(t, 0) + A(t, 0) = X(t, \pi) + A(t, \pi) = \ln(\sin^3 t) \text{ for the } S^1 \otimes S^2 \text{ topology,} & (4'''a) \\ X(t, 0) + A(t, 0) = X(t, \pi) + A(t, \pi) = \ln(\frac{1}{2}\sin t) \text{ for the } S^3 \text{ topology.} & (4'''b) \end{cases}$$

The starting point of the Hamiltonian formulation of these Gowdy universes is the form (1) of the metric with the function Y replaced by its value (3a) or (3b) and R by $\sin\theta \sin T$, where T is a function of t and θ increasing from 0 to π as t runs from 0 to π, i.e. in the cases of $S^1 \otimes S^2$ and S^3 topologies (respectively denoted by the indices 'a' and 'b'):

$$(g_{ij})_{\mathrm{a}} = \mathrm{diag}(\mathrm{e}^A, \mathrm{e}^X, \sin^2 T\, \mathrm{e}^{-X}), \tag{5a}$$

$$(g_{ij})_{\mathrm{b}} = \mathrm{diag}(\mathrm{e}^A, \tfrac{1}{2}\sin T\, \mathrm{e}^X, \tfrac{1}{2}\sin T\, \mathrm{e}^{-X}), \tag{5b}$$

in the Cartan frame respectively characterised by

$$\omega^1 = \mathrm{d}\theta, \qquad \omega^2 = \mathrm{d}\sigma, \qquad \omega^3 = \sin\theta\, \mathrm{d}\delta, \tag{6a}$$

$$\omega^1 = \mathrm{d}\theta, \qquad \omega^2 = 2\cos(\theta/2)\, \mathrm{d}\sigma, \qquad \omega^3 = 2\sin(\theta/2)\, \mathrm{d}\delta. \tag{6b}$$

The corresponding diagonal conjugate momenta are given in both cases by

$$(\pi^{ij})_{\mathrm{a}} = (\mathrm{e}^{-A}\pi_A, \mathrm{e}^{-X}(\pi_x + \tfrac{1}{2}\pi_T/\cos T), \mathrm{e}^X(\pi_T/\sin 2T)), \tag{7a}$$

$$(\pi^{ij})_{\mathrm{b}} = (\mathrm{e}^{-A}\pi_A, \mathrm{e}^{-X}(\pi_T/\cos T + \pi_X/\sin T), \mathrm{e}^X(\pi_T/\cos T - \pi_X/\sin T)), \tag{7b}$$

where π_T, π_A, π_X are the momenta conjugate to T, A and X respectively.

The action for the gravitational field can then be written as (units are chosen so that $c = 1$ and $16\pi G = 1$)

$$S = \int (\pi_T \dot{T} + \pi_X \dot{X} + \pi_A \dot{A} - N\mathscr{H} - N_i \mathscr{H}^i)\, \mathrm{d}t \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 \tag{8}$$

where the integration has to be extended to the whole universe. N and N_i are the usual lapse and shift functions and $\mathscr{H}$ and $\mathscr{H}^i$ the super-Hamiltonian and super-momenta vacuum constraints. The expressions of $\mathscr{H}$ and $\mathscr{H}^i$, calculated with the help of a Reduce 2 algebraic program (cf Moussiaux *et al* 1982), are the following:

$$\bar{\mathscr{H}}_{\mathrm{a}} = \mathrm{e}^{A/2}\mathscr{H}_a = (1/2 \sin T)(\pi_A^2 - 2\pi_A\pi_T \tan T - 2\pi_A\pi_X) + \tfrac{1}{2}\sin T[X'^2 - 2\cot\theta(A'+X') - 4] + \cos T[2T'' - T'(A + x' - 4\cot\theta)] - 2T'^2 \sin T, \tag{9a}$$

$$\bar{\mathscr{H}}^1_{\mathrm{a}} = \mathrm{e}^{A}\mathscr{H}^1_a = \pi_T T' + \pi_A A' + \pi_X X' + (1/\sin\theta)[\pi_T \tan \mathrm{T} \cos\theta - 2(\pi_A \sin\theta)'], \tag{9b}$$

$$\mathscr{H}^2_{\mathrm{a}} = \mathscr{H}^3_{\mathrm{a}} = 0; \tag{9c}$$

$$\bar{\mathscr{H}}_{\mathrm{b}} = \mathrm{e}^{A/2}\mathscr{H}_b = (\sin T)^{-1}(\pi_A^2 - 2\pi_A\pi_T \tan T + \pi_X^2) + \tfrac{1}{4}\sin T[X'^2 - (2/\sin\theta)(\cos\theta A' + X') - 3] + \tfrac{1}{2}\cos T[2T'' - T'(A' - 3\cot\theta)] + T'^2(1 + 3\sin^2 \mathrm{T})/\sin T, \tag{10a}$$

$$\bar{\mathscr{H}}^1_{\mathrm{b}} = \mathrm{e}^{A}\mathscr{H}^1_{\mathrm{b}} = \pi_T T' + \pi_A A' + \pi_X X' + (\sin\theta)^{-1}[\pi_T \tan T \cos\theta - \pi_X - 2(\pi_A \sin\theta)'], \tag{10b}$$

$$\mathscr{H}^2_{\mathrm{b}} = \mathscr{H}^3_{\mathrm{b}} = 0. \tag{10c}$$

Integration in (8) on σ and δ gives for the action:

$$S = 4\pi^2\varepsilon \int_0^\pi \mathrm{d}t \int_0^\pi \mathrm{d}\theta \sin\theta(\pi_T\dot{T} + \pi_A\dot{A} + \pi_X\dot{X} + \mathscr{H}_{\mathrm{D}}) \tag{11}$$

where $\mathscr{H}_{\mathrm{D}} = \bar{N}\bar{\mathscr{H}} + \bar{N}_1\bar{\mathscr{H}}^1$ is Dirac's Hamiltonian density; $\bar{N} = N\,\mathrm{e}^{-A/2}$ and $\bar{N}_1 = N_1\,\mathrm{e}^{-A}$ and $\varepsilon_a = 1$ and $\varepsilon_b = 2$.

The ADM reduction process consists of three steps.

(a) Perform the canonical transformation

$$(T, A, X; \pi_T, \pi_A, \pi_X) \to (T, B, X; \bar{\pi}_T, \pi_B, \pi_X) \tag{12}$$

defined by

$$\pi_A = (-1/\varepsilon \sin\theta)(\cos T B)', \qquad A' = (-\varepsilon \sin\theta/\cos T)\pi_B, \tag{13a, b}$$

$$\pi_T = \bar{\pi}_T + B\pi_B \tan T. \tag{13c}$$

(b) Impose the two coordinate conditions

$$T(t, \theta) = t, \qquad B(t, \theta) = -\cos\theta. \tag{14a, b}$$

(c) Solve the constraint equations $\bar{\mathscr{H}} = 0$ and $\bar{\mathscr{H}}^1 = 0$, respectively for $\bar{\pi}_T$ and π_B, with $\mathscr{H}$ and $\mathscr{H}^1$ respectively given by (9*a*) and (9*b*) or (10*a*) and (10*b*).

The final result is the action integral

$$S = 4\pi^2\varepsilon \int_0^\pi \mathrm{d}t \int_0^\pi \mathrm{d}\theta \sin\theta(\pi_X\dot{X} - \mathscr{H}_{\mathrm{ADM}}) \tag{15}$$

where $\mathscr{H}_{\mathrm{ADM}}$ is the reduced ADM Hamiltonian density

$$(\mathscr{H}_{\mathrm{ADM}})_{\mathrm{a}} = (1/2 \sin t)(\pi_X^2 + 2\pi_X \cos t) + \tfrac{1}{2}\sin t(X'^2 - 2X), \tag{16a}$$

$$(\mathscr{H}_{\mathrm{ADM}})_{\mathrm{b}} = (1/\sin t)\pi_X^2 + \tfrac{1}{4}\sin t\, X'^2. \tag{16b}$$

These expressions have been obtained by discarding integrated terms which can be shown to vanish due to the imposition of condition (4*a*) or (4*b*).

The final form of the constraint $\bar{\mathscr{H}}^1 = 0$, which serves to evaluate the momentum conjugate to $B = -\cos\theta$, i.e. π_B, leads to the relations

$$\pi_B(\sin^2\theta - \sin^2 t) + \cos^2 t \cos\theta(\pi_X X' \tan\theta + \bar{\pi}_T \tan t + 2\cos t) = 0 \qquad (\text{case } S^1 \otimes S^2), \quad (17a)$$

$$\pi_B(\sin^2\theta - \sin^2 t) + \cos^2 t \cos\theta(\pi_X X' \tan\theta - \pi_X \cos\theta + \bar{\pi}_T \tan t + \cos t) = 0 \qquad (\text{case } S^3), \quad (17b)$$

where $\bar{\pi}_T$ has to be expressed in terms of π_X, X and X' by using equation (9*a*) or (10*a*) in which the two first steps of the reduction process are explicitly performed.

The canonical equations corresponding to the fully reduced Hamiltonian are given by

$$\dot{X} = (\sin\theta)^{-1}(\delta/\delta\pi_X)(\sin\theta\mathscr{H}_{\mathrm{ADM}}), \qquad \dot{\pi}_X = (\sin\theta)^{-1}(\delta/\delta X)(\sin\theta\mathscr{H}_{\mathrm{ADM}}), \quad (18a, b)$$

where $\delta/\delta f$ denotes the functional derivative with respect to f. In particular, (18*b*) takes the following form for both types of Gowdy models:

$$(1/\sin t)(\sin t\dot{X})^{\cdot} - (1/\sin\theta)(\sin\theta X')' = 0 \qquad (19)$$

which is equivalent to a combination of Einstein's field equations (cf Gowdy 1974). This equivalence shows the coherence of the reduction process used above.

Turning now to the problem of the search for exact solutions of Einstein's field equations for these Gowdy models, we note first of all that the most general regular solution of equation (19) can be written as

$$X(t, \theta) = \sum_{n=0}^{\infty} P_n(\cos\theta)(\alpha_n Q_n(\cos t) + \beta_n P_n(\cos t)) \qquad (20)$$

where P_n and Q_n are Legendre functions of the first and second kind and α_n and β_n are adjustable constant coefficients. The possible values of these coefficients are constrained by the relations

$$\left(\sum \alpha_{2n}, \sum \alpha_{2n+1}\right) = (\pm 2, \gamma) \qquad \text{or} \qquad (0, \gamma \pm 2) \qquad \text{with } \gamma = \varepsilon - 1 \qquad (21a, b)$$

which are deduced from the constraint equations (17*a*) and (17*b*), with $\sin^2 t - \sin^2\theta = 0$; the constraints (21*a*) are in fact equivalent to Gowdy's matching conditions.

The constraint equations (17*a*) and (17*b*) can then be solved for π_B, thus for A' (cf (13*b*)). The resulting expression is then integrated to give explicitly $A(t, \theta)$, by taking into account (21*a*) as well as (4‴*a*) or (4‴*b*), these last conditions giving rise, as shown below, to another constraint on the constants α_n and β_n.

However, in the most general case ($n \to \infty$) this resolution process appears rather intricate to apply. An explicit complete solution has been obtained in the case of the three first coupled modes ($n = 0, 1, 2$); this solution is presented, for both types of Gowdy models, in the appendix. We have checked that it is consistent with the solution built by Gowdy (1975) by solving directly Einstein's field equations, after the corrections of a certain number of mistakes appearing in Gowdy's work. In fact, Gowdy has proposed a general solution for the field equations; however, it is so complicated and intricate that its general expression cannot reasonably be very useful;

the scheme used here and based on the Hamiltonian formalism could, in our opinion, lead more easily to the explicit form of the exact solution at any order.

In any case, we have been able to derive the following explicit form for the most general exact solution in the neighbourhood of the initial ($t=0$) and final ($t=\pi$) singularities of these Gowdy models; the asymptotic expressions of the components of the metric tensor limited to the first order in t (or $(\pi-t)$) are the following:

$$g_{11}=-g_{\infty}\sim f_1(\theta)(t/2)^{(\chi-2+\gamma)(\chi+\gamma)/2}, \tag{22a}$$

$$g_{22}\sim f_2(\theta)(t/2)^{\chi+\gamma}, \qquad g_{33}\sim (f_2(\theta))^{-1}(t/2)^{2-\chi-\gamma}, \tag{22b, c}$$

(where t has to be replaced by $\pi-t$ in the neighbourhood of the final singularity).

$f_1(\theta)$ and $f_2(\theta)$ are known functions of $\cos\theta$ only and χ is defined by

$$\chi=-\sum_{n=0}^{\infty}\tau^{n+1}\alpha_n P_n(\cos\theta) \tag{23}$$

where $\tau=+1$ (respectively -1) in the neighbourhood of $t=0$ ($t=\pi$).

As a byproduct of this derivation of the asymptotic behaviour of the metric of these Gowdy universes, we have derived the general form of the second restriction on the constants α_n and β_n, implied by equations ($4'''a$) and ($4'''b$):

$$\sum_{n=1}^{\infty}\beta_{2n}\sum_{m=1}^{n}\alpha_{2m-1}+\sum_{n=0}^{\infty}\beta_{2n+1}\sum_{m=0}^{n}\alpha_{2n}=\gamma\sum_{n=0}^{\infty}\beta_{2n} \tag{24}$$

(the restricted form of these general constraints for the modes $n=0$, 1, 2 appears explicitly in the appendix).

The principal features of the singularities of these models can be derived from the asymptotic expressions ($22a$)–($22c$).

First of all, the curvature invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ can be shown to become infinite for $t\to 0$ or π, implying that the initial and final singularities of these Gowdy universes are scalar curvature singularities in Ellis and Schmidt's (1977) sense. The spatial volume of these universes can also be shown to vanish at both singularities.

These singularities are inhomogeneous and for a given θ, two types of local behaviour are possible: one of the g_{ii} tends to infinity while the two others vanish for $t\to 0$ or π (cigar-type behaviour characteristic of spatially homogeneous models), or two g_{ii} have a finite limit value different from zero while the third one tends to zero for $t\to 0$ or π (pancake-like behaviour characteristic of spatially homogeneous models).

The asymptotic solutions ($22a$)–($22c$) display, in fact, a generalised Kasner-like behaviour typical of a general cosmological solution with a time singularity (cf Belinskii *et al* 1970). The sign of the exponents of t in solution ($22a$)–($22c$) can change with θ and so the direction of the Kasner axis will change with θ; an alternation of 'cigar' and 'pancake' will thus appear along the θ-direction.

It is also interesting to remark that the initial and final singularities will not be precisely identical, in the sense that for the same value of θ, the initial singularity can be of the 'cigar' type while the final one would be of the 'pancake' type, and if they are both of the 'cigar' type, the direction of the Kasner axis will not necessarily be the same.

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Appendix 1. Exact solutions for the Gowdy models (modes 0, 1, 2)

With the convention $O_n = \ln \tan(t/2) - \beta_n/\alpha_n$, the solution for X, for both types of models, can be written as

$$X = X_0 + X_1 + X_2$$

with

$$X_0 = -\alpha_0 O_0, \qquad X_1 = -\alpha_1 \cos\theta\,(\cos t\, O_1 + 1),$$
$$X_2 = (-\alpha_2/4)(3\cos^2\theta - 1)[(3\cos^2 t - 1)O_2 + 3\cos t].$$

In the case $S^1 \otimes S^2$, the expression of the function A corresponding to the function X is

$$A = A^* - X + \ln(\sin^2 t)$$

with A^* given by

$$\begin{aligned} A^* = {} & \tfrac{1}{4}\alpha_1^2 \sin^2\theta(\sin^2 t O_1^2 - 2\cos t\, O_1 - 1) \\ & + \tfrac{3}{32}\alpha_2^2 \sin^2\theta\{-3\sin^2 t O_2^2[\sin^2\theta(8 - 9\sin^2 t) - 8\cos^2 t] \\ & + 2\cos t O_2[3\sin^2\theta(2 - 9\sin^2 t) + 24\sin^2 t - (8/\alpha_2)(\alpha_0 + \alpha_2)] \\ & + 3\sin^2\theta(5 - 9\sin^2 t) + 24\sin^2 t - (16/\alpha_2)(\alpha_0 + \alpha_2)\} \\ & + \tfrac{1}{2}\alpha_1\alpha_2 \cos\theta \sin^2\theta[3\sin^2 t\cos t\, O_1 O_2 - (1 - 3\sin^2 t)O_1 \\ & + (2 - 3\sin^2 t)O_2 - 3\cos t] \end{aligned}$$

and with the following conditions on α_n and β_n:

$$(\alpha_0 + \alpha_2, \alpha_1) = (\pm 2, 0) \text{ or } (0, \pm 2) \qquad \text{and} \qquad \beta_2\alpha_1 + \beta_1\alpha_0 = 0.$$

In the case S^3, the function A is

$$A = A^* - \cos\theta\,(X_0 + X_2(t, 0)) - X_1(t, 0) + \ln(\tfrac{1}{2}\sin t)$$

with

$$(\alpha_0 + \alpha_2, \alpha_1) - (\pm 2, 1) \text{ or } (0, 1 \pm 2) \qquad \text{and} \qquad \beta_2(\alpha_1 - 1) + \beta_1\alpha_0 = \beta_0 = 0.$$

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